Published by IOP Publishing for SISSA
Received: *October 13, 2008*
Accepted: *November 3, 2008*
Published: *November 20, 2008*

Elliptic recurrence representation of the $N = 1$ superconformal blocks in the Ramond sector

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Abstract: The structure of the 4-point $N = 1$ super-conformal blocks in the Ramond sector is analyzed. The elliptic recursion relations for these blocks are derived.

Keywords: Conformal and W Symmetry, Conformal Field Models in String Theory.

JHEP11(2008)060

Contents

1. Introduction

Correlation functions in any 2-dimensional CFT can be expressed as sums (or integrals) of three-point coupling constants and some universal, model independent functions called conformal blocks [1]. Even in a simple case of the 4-point conformal block its direct calculation is prohibitively complicated. Efficient recursive methods of an approximate, analytic determination of a general 4-point conformal block has been pioneered long time ago by Al. Zamolodchikov [2 − 4]. His method was used for instance in checking the conformal bootstrap in the Liouville field theory with the Dorn, Otto, Zamolodchikov and Zamolodchikov coupling constants [5], in a study of the $c \to 1$ limit of minimal models [6] or in obtaining new results in the classical geometry of hyperbolic surfaces [7].

Recently a recursion representation has been worked out for the super-conformal blocks related to the Neveu-Schwarz algebra [8 − 11]. The so called elliptic recursion was conjectured in [10] for one type of NS blocks and used for a numerical verification of the consistency of $N = 1$ super-Liouville field theory. An extension of this method to another type of NS blocks was proposed in [11] where also further numerical support for the consistency of the $N = 1$ super-Liouville theory was presented. A comprehensive derivation of the elliptic recursion for all types of NS blocks was given in [12].

In the present paper we address the problem of the elliptic recurrence for conformal blocks in the Ramond sector of N=1 SCFT. We restrict ourselves to the class of SCFT models with the tensor product $\mathcal{R} \otimes \bar{\mathcal{R}}$ of the left $\mathcal{R}$ and the right $\bar{\mathcal{R}}$ Ramond algebras extended by the common (for the left and the right sector) parity operator $(-1)^F$. We shall also consider only 4-point blocks with Ramond external states which correspond to factorization on Neveu-Schwarz states.

There are some features of the Ramond sector which make a standard analysis of conformal blocks more difficult than in the NS and the bosonic cases.[1] First of all the structure of 3-point conformal blocks is more complicated. Since the correlation functions of one fermionic current, two Ramond and one Neveu-Schwarz fields are double valued the standard contour deformation arguments do not work. One way to avoid this problem has been proposed in the early days of the 2-D SCFT [13] (see also [14] for more detailed analysis). In the present paper we follow essentially the same approach considering single valued functions

$$\langle \phi(\infty,\infty) S(w) R_2(z,\bar{z}) R_1(0,0) \rangle \sqrt{(w-z)w}$$

and expressing contour integrals around a location of each field in terms of fields excitations. This leads to the Ward identities (4.1), (4.2) which determine the 3-point blocks up to four independent constants. Fortunately in spite of their complicated form one can derive all properties of the 3-point blocks required for the derivation of the elliptic recurrence formula. Let us mention that the so called x-expansion seems to be available in the present case only via the elliptic recurrence.

The second complication arises from the fact that the tensor product of the left and the right chiral structures has to be reduced to an irreducible representation of $\mathcal{R} \otimes \bar{\mathcal{R}}$ extended by $(-1)^F$. This reduction is responsible for the reduction of eight independent constants hidden in 3-point blocks into two independent structure constants of the Ramond sector. It also determines the representation of the Ramond fields in terms of chiral vertex operators and suggests a convenient basis of the 3-point blocks for which this representation is diagonal.

Once the structure and properties of the 3-point blocks are clarified the definition of 4-point conformal blocks is straightforward. As in the case of the NS sector one gets four even and four odd conformal blocks. With the appropriate definition at hand one can follow Zomolodchikov's method of the derivation of the elliptic recurrence [4]. As in the NS case [12] the arguments concerning large intermediate weight Δ asymptotic are based on the quasiclassical limit of the path integral representation of the Liouville theory. Regular terms of elliptic blocks can be calculated from the $\hat{c} = 1$ conformal block with $\Delta_i = \frac{1}{16}$ Ramond external states and an arbitrary intermediate weight. An explicit formula for this blocks can be obtained using the techniques of the chiral superscalar model [17].

The organization of the paper is as follows. In section 2 we present our notation and basic properties of the 3-point functions in the Ramond sector [15]. Section 3 is devoted to the basic structure of the highest weight chiral module for the Ramond algebra extended by the chiral parity operator [16]. The reduction of the tensor product of the left and the right modules to an irreducible representation of $\mathcal{R} \otimes \bar{\mathcal{R}}$ extended by $(-1)^F$ is briefly described. In section 4 we use the Ward identities to determine the properties of the 3-point conformal blocks. This section contains the main results of the present paper and paves a way for an appropriate definition of the conformal blocks. In section 5 we define 4-point conformal blocks and analyze their analytic properties as functions of the intermediate weight. The

[1]The structure of the 3-point conformal blocks in the Ramond sector is scarcely discussed in literature [13, 14]. We are not aware of any discussion on the 4-point blocks with external Ramond states.

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main result of this section is a calculation of the residua at singular weights. As a side result we obtained a universal property of the Ramond structure constants $C^{\pm}$ (2.7) in a general $N=1$ SCFT. If the even fusion rules (5.13) are satisfied $C^{+}=-C^{-}$, while for the odd fusion rules (5.14) one has $C^{+}=C^{-}$. In section 6 we discus the large Δ asymptotic of the conformal blocks and derive the elliptic recurrence formula. The regular terms of elliptic blocks are also calculated.

There are some problems which are natural continuation of the present work. First of all one should extend the analysis to the blocks related to the factorization on Ramond states. This can be done with the techniques developed in the present paper. The second possible topic is to extend the elliptic recurrence methods to the $N=2$ SCFT. Let us also mention that one can use our results to check consistency of the $N=1$ super Liouville theory [18] or its $\hat{c}\to 1$ limit [19].

2. Three-point correlation functions of the Ramond sector

The superconformal symmetry is generated by a pair of holomorphic currents $T(z)$, $S(z)$ (and their anti-holomorphic counterparts $\overline{T}(\bar{z})$, $\overline{S}(\bar{z})$) satisfying the OPE-s

$$\begin{aligned} T(z)T(0) &= \frac{c}{2z^4}+\frac{2}{z^2}T(0)+\frac{1}{z}\partial T(0)+\cdots, \\ T(z)S(0) &= \frac{3}{2z^2}S(0)+\frac{1}{z}\partial S(0)+\cdots, \\ S(z)S(0) &= \frac{2c}{3z^3}+\frac{2}{z}T(0)+\cdots. \end{aligned} \tag{2.1}$$

The space of fields of superconformal field theory (hereafter SCFT) decomposes onto the space of the Neveu-Schwarz (or NS for short) fields $\phi_{\rm NS}$ local with respect to $S(z)$, and the space of the Ramond fields R with the property that any correlation function of the form $\langle S(z)R(z_1,\bar{z}_1)\ldots\rangle$ changes the sing upon analytic continuation in z around the point $z=z_1$. This property implies the following form of the OPE:

$$S(z)R(0,0) \;=\; \sum_{m\in\mathbb{Z}} z^{m-\frac{3}{2}}S_{-m}R(0,0).$$

Together with the usual Virasoro generators L_n defined by

$$T(z)R(0,0) \;=\; \sum_{n\in\mathbb{Z}} z^{n-2}L_{-n}R(0,0),$$

S_k form the Ramond algebra determined by (2.1),

$$\begin{aligned} [L_m,L_n] &= (m-n)L_{m+n}+\frac{c}{12}m\left(m^2-1\right)\delta_{m+n}, \\ [L_m,S_n] &= \frac{m-2n}{2}S_{m+n}, \\ \{S_m,S_n\} &= 2L_{m+n}+\frac{c}{3}\left(m^2-\frac{1}{4}\right)\delta_{m+n}. \end{aligned} \tag{2.2}$$

In the space of all R fields there exist "super-primary" fields $R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w})$ with the conformal weights $\Delta, \bar{\Delta}$, which satisfy the OPE's[2]

$$\begin{aligned} T(z)R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{\Delta}{(z-w)^2}R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) + \frac{1}{z-w}\partial R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}), \\ \bar{T}(\bar{z})R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{\bar{\Delta}}{(\bar{z}-\bar{w})^2}R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} S(z)R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{i\beta e^{\mp i\frac{\pi}{4}}}{(z-w)^{\frac{3}{2}}}R^{\mp}_{\Delta,\bar{\Delta}}(w,\bar{w}), \\ \bar{S}(\bar{z})R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{-i\bar{\beta} e^{\pm i\frac{\pi}{4}}}{(\bar{z}-\bar{w})^{\frac{3}{2}}}R^{\mp}_{\Delta,\bar{\Delta}}(w,\bar{w}), \end{aligned} \tag{2.4}$$

where $\beta, \bar{\beta}$ are related to the conformal weights by

$$\Delta = \frac{c}{24} - \beta^2, \qquad \bar{\Delta} = \frac{c}{24} - \bar{\beta}^2.$$

Using projective transformations one can express three-point function involving two R primary fields and one NS primary superfield

$$\Phi_3(z,\theta;\bar{z},\bar{\theta}) = \varphi(z,\bar{z}) + \theta\psi(z,\bar{z}) + \bar{\theta}\bar{\psi}(z,\bar{z}) + i\theta\bar{\theta}\widetilde{\varphi}(z,\bar{z})$$

in the form

$$\begin{aligned} \Big\langle \Phi_3(z_3,\theta_3;\bar{z}_3,\bar{\theta}_3) R_2^{(\epsilon_2)}(z_2,\bar{z}_2) R_1^{(\epsilon_1)}(z_1,\bar{z}_1) \Big\rangle &= z_{32}^{\gamma_1}\bar{z}_{32}^{\bar{\gamma}_1}\, z_{31}^{\gamma_2}\bar{z}_{31}^{\bar{\gamma}_2}\, z_{21}^{\gamma_3}\bar{z}_{21}^{\bar{\gamma}_3} \\ &\times \Bigg[\delta_{\epsilon_1,\epsilon_2} \left(C^{\epsilon_1}_{321} + \widetilde{C}^{\epsilon_1}_{321} \left| \frac{z_{31}z_{32}}{z_{12}} \right| i\theta_3\bar{\theta}_3 \right) \qquad (2.5) \\ &\quad + \delta_{\epsilon_1,-\epsilon_2} \left(D^{\epsilon_1}_{321} \left(\frac{z_{31}z_{32}}{z_{12}} \right)^{\frac{1}{2}} \theta_3 + \bar{D}^{\epsilon_1}_{321} \left(\frac{\bar{z}_{31}\bar{z}_{32}}{\bar{z}_{12}} \right)^{\frac{1}{2}} \bar{\theta}_3 \right) \Bigg] \qquad (2.6) \end{aligned}$$

where $\gamma_1 = \Delta_1 - \Delta_2 - \Delta_3$, $z_{12} = z_1 - z_2$ etc. and

$$C^{\pm}_{321} = \Big\langle \varphi_3(\infty,\infty) R_2^{\pm}(1,1) R_1^{\pm}(0,0) \Big\rangle, \tag{2.7}$$

$$\widetilde{C}^{\pm}_{321} = \Big\langle \widetilde{\varphi}_3(\infty,\infty) R_2^{\pm}(1,1) R_1^{\pm}(0,0) \Big\rangle, \tag{2.8}$$

$$D^{\pm}_{321} = \Big\langle \psi_3(\infty,\infty) R_2^{\pm}(1,1) R_1^{\mp}(0,0) \Big\rangle, \tag{2.9}$$

$$\bar{D}^{\pm}_{321} = \Big\langle \bar{\psi}_3(\infty,\infty) R_2^{\pm}(1,1) R_1^{\mp}(0,0) \Big\rangle. \tag{2.10}$$

Due to the Ward identities

$$\begin{aligned} &\Big\langle S_{-\frac{1}{2}}\varphi_3(\infty,\infty) R_2^{\epsilon}(1,1) R_1^{\epsilon'}(0,0) \Big\rangle \\ &\quad = \Big\langle \varphi_3(\infty,\infty) S_0 R_2^{\epsilon}(1,1) R_1^{\epsilon'}(0,0) \Big\rangle + i\epsilon \Big\langle \varphi_3(\infty,\infty) R_2^{\epsilon}(1,1) S_0 R_1^{\epsilon'}(0,0) \Big\rangle, \end{aligned}$$

[2]Following [15] we chose the "symmetric" convention for $\pm$ components of the Ramond fields.

$$\begin{aligned}&\left\langle \bar{S}_{-\frac{1}{2}}\varphi_3(\infty,\infty)R_2^{\epsilon}(1,1)R_1^{\epsilon'}(0,0)\right\rangle\\&\quad= \left\langle \varphi_3(\infty,\infty)\bar{S}_0R_2^{\epsilon}(1,1)R_1^{\epsilon'}(0,0)\right\rangle - i\epsilon\left\langle \varphi_3(\infty,\infty)R_2^{\epsilon}(1,1)\bar{S}_0R_1^{\epsilon'}(0,0)\right\rangle,\end{aligned}$$

only two of these structure constants are independent:

$$\begin{aligned}\widetilde{C}^{\pm}_{321} &= \mp i\left[(\bar{\beta}_1\beta_1+\bar{\beta}_2\beta_2)C^{\pm}_{321} - (\bar{\beta}_1\beta_2+\bar{\beta}_2\beta_1)C^{\mp}_{321}\right],\\ D^{\pm}_{321} &= i\mathrm{e}^{\pm i\frac{\pi}{4}}\left[\beta_2C^{\mp}_{321}+\beta_1C^{\pm}_{321}\right],\\ \bar{D}^{\pm}_{321} &= -i\mathrm{e}^{\mp i\frac{\pi}{4}}\left[\bar{\beta}_2C^{\mp}_{321}+\bar{\beta}_1C^{\pm}_{321}\right].\end{aligned}\tag{2.11}$$

3. R supermodule

In SCFT one usually works with the Ramond algebra (2.2) extended by the fermion parity operator $(-1)^F$:

$$[(-1)^F, L_m] = \{(-1)^F, S_n\} = 0, \qquad m,n\in\mathbb{Z}.$$

Let w^+_Δ be the highest weight state with respect to the extended Ramond algebra (2.2)

$$L_0w^+_\Delta = \Delta w^+_\Delta, \quad (-1)^F w^+_\Delta = w^+_\Delta, \quad L_mw^+_\Delta = S_nw^+_\Delta = 0, \quad m,n\in\mathbb{N}, \tag{3.1}$$

where $\mathbb{N}$ is the set of positive integers. We denote by $\mathcal{W}^f_\Delta$ the free vector space generated by all vectors of the form

$$w^+_{\Delta,KM} = S_{-K}L_{-M}w^+_\Delta \equiv S_{-k_i}\dots S_{-k_1}L_{-m_j}\dots L_{-m_1}w^+_\Delta, \tag{3.2}$$

where $K=\{k_1,k_2,\dots,k_i\}\subset\mathbb{N}\cup\{0\}$ and $M=\{m_1,m_2,\dots,m_j\}\subset\mathbb{N}$ are arbitrary ordered sets of indices

$$k_i<\dots<k_2<k_1, \qquad m_j\le\dots\le m_2\le m_1,$$

such that $|K|+|M| = k_1+\dots+k_i+m_1+\dots+m_j = f$.

The $\mathbb{Z}$-graded representation of the extended Ramond algebra determined on the space

$$\mathcal{W}_\Delta = \bigoplus_{f\in\mathbb{N}\cup\{0\}}\mathcal{W}^f_\Delta$$

by the relations (2.2) and (3.1) is called the R supermodule of the highest weight Δ and the central charge c (in order to simplify the notation we omit the subscript c at $\mathcal{W}$). Each $\mathcal{W}^f_\Delta$ is an eigenspace of L_0 with the eigenvalue $\Delta+f$. The space $\mathcal{W}_\Delta$ has also a natural $\mathbb{Z}_2$-grading:

$$\mathcal{W}_\Delta = \mathcal{W}^+_\Delta\oplus\mathcal{W}^-_\Delta, \quad \mathcal{W}^+_\Delta = \bigoplus_{f\in\mathbb{N}\cup\{0\}}\mathcal{W}^{f+}_\Delta, \quad \mathcal{W}^-_\Delta = \bigoplus_{f\in\mathbb{N}\cup\{0\}}\mathcal{W}^{f-}_\Delta,$$

where $\mathcal{W}^{f\pm}_\Delta$ are common eigenspaces of the operators $L_0,(-1)^F$. Note that the subspaces $\mathcal{W}^{0+}_\Delta,\mathcal{W}^{0-}_\Delta$ are 1-dimensional except the case $\Delta=\frac{c}{24}$ where $\mathcal{W}^{0-}_\Delta=\{0\}$.

A nonzero element $\chi\in\mathcal{W}^f_\Delta$ of degree f is called a singular vector if it satisfies the highest weight conditions (3.1) with $L_0\chi=(\Delta+f)\chi$. It generates its own R supermodule $\mathcal{W}_{\Delta+f}$ which is a submodule of $\mathcal{W}_\Delta$.

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The analysis of singular vectors can be facilitated by introducing a symmetric bilinear form $\langle .\,,.\rangle_{c,\Delta}$ on $\mathcal{W}_\Delta$ uniquely determined by the relations $\langle w_\Delta, w_\Delta\rangle = 1$, $\langle w_\Delta, S_0\, w_\Delta\rangle = 0$ and $(L_m)^\dagger = L_{-m}, (S_k)^\dagger = S_{-k}$. It is block-diagonal with respect to the L_0- and $(-1)^F$-grading. We denote by $B^{f\pm}_{c,\Delta}$ the matrix of $\langle .\,,.\rangle_{c,\Delta}$ on $\mathcal{W}^{f\pm}_\Delta$ calculated in the basis (3.2):

$$\left[B^{f\pm}_{c,\Delta}\right]_{KM,LN} = \langle w^{\pm}_{\Delta,KM}, w^{\pm}_{\Delta,LN}\rangle_{c,\Delta}. \tag{3.3}$$

It is nonsingular if and only if the R supermodule $\mathcal{W}_\Delta$ does not contain singular vectors of degrees $0, 1, 2, \ldots, f$. The formula for the determinant of this matrix was conjectured by Friedan, Qiu and Shenker [13] and proven by Meurman and Rocha-Caridi [20]. For level zero it reads

$$\det B^{0+}_{c,\Delta} = 1, \qquad \det B^{0-}_{c,\Delta} = \Delta - \frac{c}{24},$$

and for higher levels

$$\det B^{f\pm}_{c,\Delta} = \left(\Delta - \frac{c}{24}\right)^{\frac{P_R(f)}{2}} \prod_{1\leqslant rs\leqslant 2f} (\Delta - \Delta_{rs})^{P_R(f-\frac{rs}{2})}, \tag{3.4}$$

where $r, s \in \mathbb{N}$, the sum $r+s$ must be odd and

$$\Delta_{rs}(c) = \frac{1}{16} - \frac{rs-1}{4} + \frac{1-r^2}{8}b^2 + \frac{1-s^2}{8}\frac{1}{b^2}, \qquad c = \frac{3}{2} + 3\left(b + \frac{1}{b}\right)^2. \tag{3.5}$$

The multiplicity of each zero is given by $P_R(f) = \dim \mathcal{W}^f_\Delta$ and can be read off from the relation

$$\sum_{f=0}^{\infty} P_R(f) q^f = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}.$$

The tensor product $\mathcal{W}_\Delta \otimes \bar{\mathcal{W}}_{\bar\Delta}$ of the left and the right R supermodules is defined as a graded tensor product of representations of $\mathbb{Z}_2$-graded algebras. This provides a representation of the direct sum $R \oplus \bar R$ of left and right Ramond algebras extended by the left $(-1)^{F_L}$ and the right $(-1)^{F_R}$ parity operators. We are usually interested in the extension of $R \otimes \bar R$ by the common parity operator

$$(-1)^F = (-1)^{F_L}(-1)^{F_R}$$

and the corresponding $\mathbb{Z}_2$-grading. For $\Delta, \bar\Delta \neq \frac{c}{24}$ an appropriate representation can be easily obtained restricting the action of $R \otimes \bar R$ and $(-1)^F$ to an invariant subspace $\mathcal{W}_{\Delta,\bar\Delta} \subset \mathcal{W}_\Delta \otimes \overline{\mathcal{W}}_{\bar\Delta}$ generated by the vectors

$$\begin{aligned} w^+_{\Delta,\bar\Delta} &= \frac{1}{\sqrt{2}}\left(w^+_\Delta \otimes w^+_{\bar\Delta} - i\, w^-_\Delta \otimes w^-_{\bar\Delta}\right), \\ w^-_{\Delta,\bar\Delta} &= \frac{1}{\sqrt{2}}\left(w^+_\Delta \otimes w^-_{\bar\Delta} \;+\; w^-_\Delta \otimes w^+_{\bar\Delta}\right), \end{aligned} \tag{3.6}$$

where $w^-_\Delta = \frac{e^{i\frac{\pi}{4}}}{\beta} S_0 w^+_\Delta$ and $w^-_{\bar\Delta} = \frac{e^{-i\frac{\pi}{4}}}{\bar\beta} \bar S_0 w^+_{\bar\Delta}$. We shall call it a "small" representation.

The choice of basis (3.6) in the zero level subspace $\mathcal{W}^0_{\Delta,\bar\Delta}$ corresponds to our choice of the Ramond fields (2.4)

$$S_0 w^{\pm}_{\Delta,\bar\Delta} = i\beta e^{\mp i\frac{\pi}{4}} w^{\mp}_{\Delta,\bar\Delta}, \qquad \bar S_0 w^{\pm}_{\Delta,\bar\Delta} = -i\bar\beta e^{\pm i\frac{\pi}{4}} w^{\mp}_{\Delta,\bar\Delta}. \tag{3.7}$$

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4. Three-point conformal blocks

Super-descendants $\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z})$ of the super-primary field $\varphi_{\Delta,\bar{\Delta}}(w,\bar{w}|z,\bar{z}) = \varphi_{\Delta,\bar{\Delta}}(z,\bar{z})$ are defined by the relations:

$$
\begin{aligned}
\varphi_{\Delta,\bar{\Delta}}(L_{-m}\xi,\bar{\xi}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{1-m}T(w)\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}), && m \in \mathbb{N},\\
\varphi_{\Delta,\bar{\Delta}}(S_{-k}\xi,\bar{\xi}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}-k}S(w)\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}), && k \in \mathbb{N}-\tfrac{1}{2},
\end{aligned}
$$

and by analogous formulae for the Ramond sector

$$
\begin{aligned}
R_{\Delta,\bar{\Delta}}(L_{-m}\eta,\bar{\eta}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{1-m}T(w)R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}), && m \in \mathbb{N},\\
R_{\Delta,\bar{\Delta}}(S_{-k}\eta,\bar{\eta}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}-k}S(w)R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}), && k \in \mathbb{N}.
\end{aligned}
$$

Using conformal Ward identities one can express an arbitrary correlator of three descendants as a sum of terms which can be further factorized onto a holomorphic and anti-holomorphic parts. Consider for instance a correlator of two arbitrary excited Ramond fields $R^{\epsilon}(z,\bar{z}), R^{\epsilon'}(0,0)$ of definite parities ϵ,ϵ' and an arbitrary NS excited field $\varphi(\infty,\infty)$. Due to a square root singularity the Ward identities are more complicated than in the NS sector. They read:

$$
\begin{aligned}
\sum_{p=0}^{\infty} \binom{n+\frac{1}{2}}{p}\; x^{n+\frac{1}{2}-p}\; &\langle \phi(\infty)(S_p R^{\epsilon})(x)R^{\epsilon'}(0)\rangle \\
&= \sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p} (-x)^p\; \langle (S_{p-n-\frac{1}{2}}\phi)(\infty)R^{\epsilon}(x)R^{\epsilon'}(0)\rangle \\
&\quad -i\epsilon \sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p} (-1)^p\; x^{\frac{1}{2}-p}\langle \phi(\infty)R^{\epsilon}(x)(S_{n+p}R^{\epsilon'})(0)\rangle,
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
\sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p}\; x^{\frac{1}{2}-p}\; &\langle \phi(\infty)(S_{p-n} R^{\epsilon})(x)R^{\epsilon'}(0)\rangle \\
&= \sum_{p=0}^{\infty} \binom{-n+\frac{1}{2}}{p} (-x)^p\; \langle (S_{p+n-\frac{1}{2}}\phi)(\infty)R^{\epsilon}(x)R^{\epsilon'}(0)\rangle \\
&\quad -i\epsilon \sum_{p=0}^{\infty} \binom{-n+\frac{1}{2}}{p} (-1)^{n+p} x^{-n+\frac{1}{2}-p}\langle \phi(\infty)R^{\epsilon}(x)(S_{p}R^{\epsilon'})(0)\rangle.
\end{aligned}
\tag{4.2}
$$

The corresponding relations hold in the anti-holomorphic sector.

Our aim in this section is to decompose a general 3-point functions onto left and right 3-point conformal blocks. The first complication arises due to the block structure of field operators

$$
\Phi = \left[\begin{array}{c|c} \Phi_{\mathrm{NN}} & 0 \\ \hline 0 & \Phi_{\mathrm{RR}} \end{array}\right], \qquad R^{\pm} = \left[\begin{array}{c|c} 0 & R^{\pm}_{\mathrm{NR}} \\ \hline R^{\pm}_{\mathrm{RN}} & 0 \end{array}\right],
$$

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with respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_{\rm NS} \oplus \mathcal{H}_{\rm R}$ of the space of states. Due to the hermitian conjugation rules:[3]

$$\Phi_{\Delta,\bar\Delta}(z,\bar z)^\dagger = \bar z^{-2\Delta} z^{-2\bar\Delta}\Phi_{\Delta,\bar\Delta}\left(\tfrac{1}{\bar z},\tfrac{1}{z}\right), \qquad R^{\pm}_{\Delta,\bar\Delta}(z,\bar z)^\dagger = \bar z^{-2\Delta} z^{-2\bar\Delta} R^{\pm}_{\Delta,\bar\Delta}\left(\tfrac{1}{\bar z},\tfrac{1}{z}\right),$$

the off-diagonal blocks of the Ramond field are related to each other:

$$\overline{\langle R(\infty) R^{\pm}_{\rm RN}(z,\bar z)\phi(0)\rangle} = \bar z^{-2\Delta} z^{-2\bar\Delta}\langle \phi(\infty) R^{\pm}_{\rm NR}(\tfrac{1}{\bar z},\tfrac{1}{z}) R(0)\rangle. \tag{4.3}$$

As in the NS case [8] we define the chiral vertex operators in terms of 3-linear forms. In the R-R sector the form

$$\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(\ldots;z) : \mathcal{W}_{\Delta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{W}_{\Delta_1} \;\mapsto\; \mathbb{C}$$

satisfies the relations[4]

$$\begin{aligned}\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(S_{-n}\eta_3,\xi_2,\eta_1;z) &= (-1)^{|\eta_1|+|\eta_3|+1}\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(\eta_3,\xi_2,S_n\eta_1;z)\\ &\quad+\sum_{k=-\frac12}^{\infty}\binom{n+\frac12}{k+\frac12} z^{n-k}\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(\eta_3,S_k\xi_2,\eta_1;z),\end{aligned}$$

$$\begin{aligned}&\sum_{p=0}^{\infty}\binom{\frac12}{p}\; z^{\frac12-p}\;\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(\eta_3,S_{p-k}\xi_2,\eta_1;z) =\\ &\qquad\sum_{p=0}^{\infty}\binom{\frac12-k}{p}(-z)^p\;\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(S_{p+k-\frac12}\eta_3,\xi_2,\eta_1;z)\\ &\qquad-(-1)^{|\eta_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac12-k}{p}(-z)^{\frac12-k-p}\varrho^{\Delta_3\Delta_2\Delta_1}_{RR}(\eta_3,\xi_2,S_p\eta_1;z)\end{aligned}$$

where $|\eta_i|$ denote parities of Ramond states. The form is determined by the relations above up to two independent constants. The appropriate constructions and derivations parallel to large extend those of the NS-NS sector and we shall skip them in the present paper.

Since the NS-R and R-NS sectors are related by the hermicity condition (4.3) we shall restrict ourselves to the NS-R one. The form (anti-linear in the left argument and linear in the central and the right ones)

$$\varrho^{\Delta_3\Delta_2\Delta_1}(\ldots;z) : \mathcal{V}_{\Delta_3}\times\mathcal{W}_{\Delta_2}\times\mathcal{W}_{\Delta_1} \;\mapsto\; \mathbb{C}$$

is defined by

$$\begin{aligned}&\sum_{p=0}^{\infty}\binom{n+\frac12}{p}\; z^{n+\frac12-p}\;\varrho^{\Delta_3\Delta_2\Delta_1}_{NR}(\xi_3,S_p\eta_2,\eta_1;z)\\ &\quad=\sum_{p=0}^{\infty}\binom{\frac12}{p}(-z)^p\;\varrho^{\Delta_3\Delta_2\Delta_1}_{NR}(S_{p-n-\frac12}\xi_3,\eta_2,\eta_1;z)\\ &\qquad-i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac12}{p}(-1)^p\; z^{\frac12-p}\varrho^{\Delta_3\Delta_2\Delta_1}_{NR}(\xi_3,\eta_2,S_{n+p}\eta_1;z),\end{aligned}$$

[3]We assume that R^+ is hermitian, then the hermicity of R^- follows from the definition of S_0R^+.

[4]The chiral Ward identities for the Virasoro generators L_n are the same in all sectors. The corresponding formulae can be found in [8]. For the sake of simplicity we suppress all the L_{-n} excitations dependence.

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$$\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\; \varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\xi_3, S_{p-n}\eta_2, \eta_1; z) \tag{4.4}$$

$$= \sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-z)^p\; \varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(S_{p+n-\frac{1}{2}}\xi_3, \eta_2, \eta_1; z)$$

$$- \; i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-1)^{n+p} z^{\frac{1}{2}-n-p}\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\xi_3, \eta_2, S_p\eta_1; z).$$

It is almost completely determined by these relations. In particular, for L_0-eingenstates, $L_0\,\xi_3 = \Delta_3(\xi_3)\xi_3,\; L_0\,\eta_i = \Delta_i(\eta_i)\,\eta_i,\; i = 1,2$ one has:

$$\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\xi_3, \eta_2, \eta_1; z) \;=\; z^{\Delta_3(\xi_3)-\Delta_2(\eta_2)-\Delta_1(\eta_1)}\; \varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\xi_3, \eta_2, \eta_1; 1). \tag{4.5}$$

In contrast to the forms in the NS-NS and R-R sectors it depends on four rather than two arbitrary constants. We define the forms $\rho_{NR}^{ij}\,; i, j \;=\; \pm$ as coefficients in front of these constants:

$$\begin{aligned}
\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\xi_3, \eta_2, \eta_1; z) \;=\; & \rho_{NR}^{++}(\xi_3, \eta_2, \eta_1; z)\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\nu_3, w_2^+, w_1^+; 1)\\
& +\rho_{NR}^{+-}(\xi_3, \eta_2, \eta_1; z)\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\nu_3, w_2^+, w_1^-; 1)\\
& +\rho_{NR}^{-+}(\xi_3, \eta_2, \eta_1; z)\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\nu_3, w_2^-, w_1^+; 1)\\
& +\rho_{NR}^{--}(\xi_3, \eta_2, \eta_1; z)\varrho_{NR}^{\Delta_3\Delta_2\Delta_1}(\nu_3, w_2^-, w_1^-; 1)\;.
\end{aligned} \tag{4.6}$$

From (4.5) one easily gets

$$\rho_{NR}^{ij}(\xi_3, \eta_2, \eta_1; z) \;=\; z^{\Delta_3(\xi_3)-\Delta_2(\eta_2)-\Delta_1(\eta_1)}\rho_{NR}^{ij}(\xi_3, \eta_2, \eta_1)$$

where $\rho_{NR}^{ij}(\xi_3, \eta_2, \eta_1) \;=\; \rho_{NR}^{ij}(\xi_3, \eta_2, \eta_1; 1)$. Analyzing the Ward identities (4.4) one can derive the relations:

$$\begin{aligned}
\rho_{NR}^{+-}(S_{-I}\nu, w_2^-, S_{-J}w_1^+) &= i\,\rho_{NR}^{--}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{-+}(S_{-I}\nu, w_2^-, S_{-J}w_1^+) &= \rho_{NR}^{++}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{++}(S_{-I}\nu, w_2^-, S_{-J}w_1^+) &= i\,\rho_{NR}^{-+}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{--}(S_{-I}\nu, w_2^-, S_{-J}w_1^+) &= \rho_{NR}^{+-}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\rho_{NR}^{+-}(S_{-I}\nu, w_2^+, S_{-J}w_1^-) &= \;\;(-1)^{|J|}\,\rho_{NR}^{++}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{-+}(S_{-I}\nu, w_2^+, S_{-J}w_1^-) &= -i\,(-1)^{|J|}\,\rho_{NR}^{--}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{++}(S_{-I}\nu, w_2^+, S_{-J}w_1^-) &= -i\,(-1)^{|J|}\,\rho_{NR}^{+-}(S_{-I}\nu, w_2^+, S_{-J}w_1^+),\\
\rho_{NR}^{--}(S_{-I}\nu, w_2^+, S_{-J}w_1^-) &= \;\;(-1)^{|J|}\,\rho_{NR}^{-+}(S_{-I}\nu, w_2^+, S_{-J}w_1^+).
\end{aligned} \tag{4.8}$$

We shall now briefly analyze how the global parity requirements along with the "small" representation reduces the number of independent constants in the matrix elements of

Ramond fields from eight to two. We shall start with non normalized chiral vertex operators $V^{\pm}_{\rm NRe}, V^{\pm}_{\rm NRo}$ defined in terms of their matrix elements by the form $\varrho^{\Delta_3\Delta_2\Delta_1}_{NR}$

$$\begin{aligned}
\langle \xi_3 | V^{\pm}_{\rm NRe}(z) | \eta_1 \rangle &= \varrho^{\Delta_3\Delta_2\Delta_1}_{NR}(\xi_3, w^{\pm}, \eta_1; z), && |\xi_3| + |\eta_1| \in 2\mathbb{N}, \\
\langle \xi_3 | V^{\pm}_{\rm NRo}(z) | \eta_1 \rangle &= \varrho^{\Delta_3\Delta_2\Delta_1}_{NR}(\xi_3, w^{\pm}, \eta_1; z), && |\xi_3| + |\eta_1| \in 2\mathbb{N}+1.
\end{aligned}$$

From the construction of the highest-weight vectors $w^{\pm}_{\Delta,\bar\Delta}$ (3.6) one may expect the following form of the Ramond fields

$$\begin{aligned}
R^{+}_{\rm NR} &= AV^{+}_{\rm NRe} \otimes \bar V^{+}_{\rm NRe} + BV^{+}_{\rm NRo} \otimes \bar V^{+}_{\rm NRo} + iBV^{-}_{\rm NRe} \otimes \bar V^{-}_{\rm NRe} - iAV^{-}_{\rm NRo} \otimes \bar V^{-}_{\rm NRo}\,, \\
R^{-}_{\rm NR} &= AV^{+}_{\rm NRe} \otimes \bar V^{-}_{\rm NRo} - BV^{+}_{\rm NRo} \otimes \bar V^{-}_{\rm NRe} + BV^{-}_{\rm NRe} \otimes \bar V^{+}_{\rm NRo} + AV^{-}_{\rm NRo} \otimes \bar V^{+}_{\rm NRe}\,,
\end{aligned}$$

where the coefficients are fixed up to A and B by the relations (2.11). The independent structure constants $C^{\pm}$ (2.7) are expressed in terms of A, B and constants hidden in the forms $\varrho^{\Delta_3\Delta_2\Delta_1}_{NR}$, $\bar\varrho^{\bar\Delta_3\bar\Delta_2\bar\Delta_1}_{NR}$ as follows:

$$\begin{aligned}
C^{+} &= A\,\varrho_{NR}(\nu, w^{+}, w^{+}; 1)\,\bar\varrho_{NR}(\nu, w^{+}, w^{+}; 1) + iB\,\varrho_{NR}(\nu, w^{-}, w^{+}; 1)\,\bar\varrho_{NR}(\nu, \bar w^{-}, w^{+}; 1) \\
&\quad + iB\,\varrho_{NR}(\nu, w^{+}, w^{-}; 1)\,\bar\varrho_{NR}(\nu, w^{+}, \bar w^{-}; 1) + A\,\varrho_{NR}(\nu, w^{-}, w^{-}; 1)\,\bar\varrho_{NR}(\nu, w^{-}, w^{-}; 1), \\
C^{-} &= A\,\varrho_{NR}(\nu, w^{+}, w^{+}; 1)\,\bar\varrho_{NR}(\nu, w^{-}, w^{-}; 1) + B\,\varrho_{NR}(\nu, w^{-}, w^{+})\,\bar\varrho_{NR}(\nu, w^{+}, w^{-}; 1) \\
&\quad - B\,\varrho_{NR}(\nu, w^{+}, w^{-}; 1)\,\bar\varrho_{NR}(\nu, w^{-}, w^{+}; 1) + A\,\varrho_{NR}(\nu, w^{-}, w^{-}; 1)\,\bar\varrho_{NR}(\nu, w^{+}, w^{+}; 1).
\end{aligned} \tag{4.9}$$

One can check using the relations (4.7), (4.8) that all matrix elements of the Ramond field $R^{\pm}_{NR}$ depend on the arbitrary constants only via the combinations above. Indeed, using the relations (4.8) one obtains

$$\begin{aligned}
&\left\langle S_{-I}\bar S_{-\bar I}\,\nu_3 \otimes \bar\nu_3 \right| \; R^{+}_{2} \; \left| S_{-J}\bar S_{-\bar J}\, w^{+}_{\Delta_1,\bar\Delta_1} \right\rangle \\
&\qquad = C^{(+)}\rho^{(+)}_{\rm NRe}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(+)}_{\rm NRe}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}) \\
&\qquad\quad + C^{(-)}\rho^{(-)}_{\rm NRe}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(-)}_{\rm NRe}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}) \\
&\left\langle S_{-I}\bar S_{-\bar I}\,\nu_3 \otimes \bar\nu_3 \right| \; R^{-}_{2} \; \left| S_{-J}\bar S_{-\bar J}\, w^{+}_{\Delta_1,\bar\Delta_1} \right\rangle \\
&\qquad = (-1)^{|J|}C^{(+)}\rho^{(+)}_{\rm NRe}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(+)}_{\rm NRo}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}) \\
&\qquad\quad - (-1)^{|J|}C^{(-)}\rho^{(-)}_{\rm NRe}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(-)}_{\rm NRo}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1})
\end{aligned}$$

for $2|I| = |J|$ and

$$\begin{aligned}
&\left\langle S_{-I}\bar S_{-\bar I}\,\nu_3 \otimes \bar\nu_3 \right| \; R^{+}_{2} \; \left| S_{-J}\bar S_{-\bar J}\, w^{+}_{\Delta_1,\bar\Delta_1} \right\rangle \\
&\qquad = -i(-1)^{|J|}C^{(+)}\rho^{(+)}_{\rm NRo}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(+)}_{\rm NRo}(\bar S_{-\bar I}\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}) \\
&\qquad\quad - i\,(-1)^{|J|}C^{(-)}\rho^{(-)}_{\rm NRo}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(-)}_{\rm NRo}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}), \\
&\left\langle S_{-I}\bar S_{-\bar I}\,\nu_3 \otimes \bar\nu_3 \right| \; R^{-}_{2} \; \left| S_{-J}\bar S_{-\bar J}\, w^{+}_{\Delta_1,\bar\Delta_1} \right\rangle \\
&\qquad = C^{(+)}\rho^{(+)}_{\rm NRo}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(+)}_{\rm NRe}(\bar S_{-\bar I}\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1}) \\
&\qquad\quad - C^{(-)}\rho^{(-)}_{\rm NRo}(S_{-I}\,\nu_3, w^{+}_{2}, S_{-J}\,w^{+}_{1})\,\bar\rho^{(-)}_{\rm NRe}(\bar S_{-\bar I}\,\bar\nu_3, w^{+}_{2}, \bar S_{-\bar J}\,\bar w^{+}_{1})
\end{aligned}$$

for $2|I| = |J| + 1$, where

$$C^{(\pm)} = \frac{C^+ \pm C^-}{2},$$
$$\rho^{(\pm)}_{\rm NRe} = \rho^{++}_{NR} \pm \rho^{--}_{NR}, \qquad \bar\rho^{(\pm)}_{\rm NRe} = \bar\rho^{++}_{NR} \pm \bar\rho^{--}_{NR},$$
$$\rho^{(\pm)}_{\rm NRo} = \rho^{+-}_{NR} \pm i\rho^{-+}_{NR}, \qquad \bar\rho^{(\pm)}_{\rm NRo} = \bar\rho^{+-}_{NR} \mp i\bar\rho^{-+}_{NR}.$$

Introducing chiral vertex operators

$$\begin{aligned}\langle \xi_3|V^{(\pm)}_{\rm NRe}(z)|\eta_1\rangle &= \rho^{(\pm)}_{\rm NRe}(\xi_3, w^+, \eta_1; z), & |\xi_3| + |\eta_1| \in 2\mathbb{N},\\ \langle \xi_3|V^{(\pm)}_{\rm NRo}(z)|\eta_1\rangle &= \rho^{(\pm)}_{\rm NRo}(\xi_3, w^+, \eta_1; z), & |\xi_3| + |\eta_1| \in 2\mathbb{N}+1,\end{aligned} \tag{4.10}$$

one thus gets

$$\begin{aligned}R^+_{\rm NR} &= C^{(+)}\left(V^{(+)}_{\rm NRe}\otimes \bar V^{(+)}_{\rm NRe} - i\,V^{(+)}_{\rm NRo}\otimes \bar V^{(+)}_{\rm NRo}\right) + C^{(-)}\left(V^{(-)}_{\rm NRe}\otimes \bar V^{(-)}_{\rm NRe} - i\,V^{(-)}_{\rm NRo}\otimes \bar V^{(-)}_{\rm NRo}\right),\\ R^-_{\rm NR} &= C^{(+)}\left(V^{(+)}_{\rm NRe}\otimes \bar V^{(+)}_{\rm NRo} + V^{(+)}_{\rm NRo}\otimes \bar V^{(+)}_{\rm NRe}\right) - C^{(-)}\left(V^{(-)}_{\rm NRe}\otimes \bar V^{(-)}_{\rm NRo} + V^{(-)}_{\rm NRo}\otimes \bar V^{(-)}_{\rm NRe}\right).\end{aligned} \tag{4.11}$$

The Ramond fields in the R-NS sector are then directly obtained from the hermicity condition $\left(R^\pm_{\rm NR}\right)^\dagger = R^\pm_{\rm RN}$ (4.3):

$$\begin{aligned}R^+_{\rm RN} &= C^{(+)}\left(V^{(+)\dagger}_{\rm NRe}\otimes \bar V^{(+)\dagger}_{\rm NRe} - iV^{(+)\dagger}_{\rm NRo}\otimes \bar V^{(+)\dagger}_{\rm NRo}\right) + C^{(-)}\left(V^{(-)\dagger}_{\rm NRe}\otimes \bar V^{(-)\dagger}_{\rm NRe} - iV^{(-)\dagger}_{\rm NRo}\otimes \bar V^{(-)\dagger}_{\rm NRo}\right),\\ R^-_{\rm RN} &= C^{(+)}\left(V^{(+)\dagger}_{\rm NRe}\otimes \bar V^{(+)\dagger}_{\rm NRo} + V^{(+)\dagger}_{\rm NRo}\otimes \bar V^{(+)\dagger}_{\rm NRe}\right) - C^{(-)}\left(V^{(-)\dagger}_{\rm NRe}\otimes \bar V^{(-)\dagger}_{\rm NRo} + V^{(-)\dagger}_{\rm NRo}\otimes \bar V^{(-)\dagger}_{\rm NRe}\right).\end{aligned}$$

Let us observe that

$$w^+_\Delta \otimes w^+_\Delta + i\,w^-_\Delta \otimes w^-_\Delta, \quad w^+_\Delta \otimes w^-_\Delta - w^-_\Delta \otimes w^+_\Delta \;\in\; \ker R^\pm_{\rm NR},$$

hence the "small representation" is an invariant subspace of the full Ramond fields $R^\pm$.

As a side remark let us mention that the chiral decomposition (4.11) can be easily extended to excited Ramond fields using central arguments of the forms $\rho^{(\pm)}_{\rm NRe}, \rho^{(\pm)}_{\rm NRo}$:

$$\begin{aligned}\langle \xi_3|V^{(\pm)}_{\rm NRe}(\eta_2)|\eta_1\rangle &= \varrho^{(\pm)}_{\rm NRe}(\xi_3, \eta_2, \eta_1), & |\xi_3| + |\eta_2| + |\eta_1| \in 2\mathbb{N},\\ \langle \xi_3|V^{(\pm)z}_{\rm NRo}(\eta_2)|\eta_1\rangle &= \varrho^{(\pm)}_{\rm NRo}(\xi_3, \eta_2, \eta_1), & |\xi_3| + |\eta_2| + |\eta_1| \in 2\mathbb{N}+1.\end{aligned} \tag{4.12}$$

Taking into account the graded tensor product structure:

$$\langle \xi_3\otimes\bar\xi_3|V^{(\pm)}_{\rm NRp}(\eta_2)\otimes \bar V^{(\pm)}_{\rm NR\bar p}(\bar\eta_2)|\eta_1\otimes\bar\eta_1\rangle = (-1)^{|p||\bar\eta_2|+|\bar p||\eta_1|}\varrho^{(\pm)}_{\rm NRp}(\xi_3,\eta_2,\eta_1)\bar\varrho^{(\pm)}_{\rm NR\bar p}(\bar\xi_3,\bar\eta_2,\bar\eta_1)$$

where $\rm p, \bar p = e, o$ and $|\rm e| = 0, |\rm o| = 1$, one gets for such extension

$$\begin{aligned}-iR^+(w^-\otimes\bar w^-) &= R^+(w^+\otimes\bar w^+),\\ R^+(w^+\otimes\bar w^-) &= R^+(w^-\otimes\bar w^+) = R^-(w^+\otimes\bar w^+).\end{aligned}$$

The forms $\rho^{(\pm)}_{\rm NRe}, \rho^{(\pm)}_{\rm NRo}$ depend on the sign of β_2 in a very simple way:

$$\begin{aligned}\rho^{(\pm)}_{\rm NRe}(S_{-I}\nu, w^-_{-\beta_2}, S_J w^+_1) &= \pm\rho^{(\mp)}_{\rm NRo}(S_{-I}\nu, w^+_{\beta_2}, S_J w^+_1),\\ \rho^{(\pm)}_{\rm NRo}(S_{-I}\nu, w^-_{-\beta_2}, S_J w^+_1) &= \pm\rho^{(\mp)}_{\rm NRe}(S_{-I}\nu, w^+_{\beta_2}, S_J w^+_1).\end{aligned} \tag{4.13}$$

This yields

$$R^\epsilon_{-\beta} = \epsilon R^\epsilon_\beta\,. \tag{4.14}$$

5. Analytic structure of 4-point conformal blocks

We shall restrict ourselves to the case of correlation functions of four Ramond fields. Te structure of the 3-point conformal blocks analyzed in the previous section suggests the following definition:

$$F^{f}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} \;=\; \sum_{|K|+|M|=|L|+|N|=f} \overline{\rho^{(\pm)}_{|f|}(\nu_{\Delta,KM}, w^{+}_{\beta_3}, w^{+}_{4})}\;\left[B^{f}_{c,\Delta}\right]^{KM,LN}\rho^{(\pm)}_{|f|}(\nu_{\Delta,LN}, w^{+}_{\beta_2}, w^{+}_{1}),$$

where $|f| = \mathrm{e}$ for $f \in \mathbb{N}$, $|f| = \mathrm{o}$ for $f \in \mathbb{N}-\frac{1}{2}$, $\nu_{\Delta,KM}$ is the standard basis in the NS Verma module $\mathcal{V}_{c,\Delta}$, and $\left[B^{f}_{c,\Delta}\right]^{KM,LN}$ denotes the inverse to the Gram matrix with respect to this basis on the level $f \in \frac{1}{2}\mathbb{N}$. One has four even:

$$\mathcal{F}^{1}_{\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}(z) \;=\; z^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{m\in\mathbb{N}} z^{m}F^{m}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}\right), \tag{5.1}$$

and four odd,

$$\mathcal{F}^{\frac{1}{2}}_{\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}(z) \;=\; z^{\Delta+\frac{1}{2}-\Delta_2-\Delta_1}\sum_{k\in\mathbb{N}-\frac{1}{2}} z^{k-\frac{1}{2}}F^{k}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}, \tag{5.2}$$

conformal blocks.

It follows from the definition of the blocks' coefficients $F^{f}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}$ that they are polynomials in β_i, and rational functions of the intermediate weight Δ and the central charge c. They can be expressed as a sum over the poles in Δ :

$$F^{f}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} \;=\; R^{f}_{c,\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} + \sum_{\substack{1<rs\leq 2f\\ r+s\in 2\mathbb{N}}} \frac{\mathcal{R}^{f}_{c,\,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}}{\Delta-\Delta_{rs}(c)}, \tag{5.3}$$

with $\Delta_{rs}(c)$ given by Kac determinant formula for NS Verma modules:

$$\Delta_{rs}(c) \;=\; -\frac{rs-1}{4}+\frac{1-r^2}{8}b^2+\frac{1-s^2}{8}\frac{1}{b^2}, \qquad c=\frac{3}{2}+3\left(b+\frac{1}{b}\right)^2. \tag{5.4}$$

The calculation of the residue at Δ_{rs} is essentially the same as in the NS case. With a suitable choice of basis in $\mathcal{V}_{\Delta}$ one gets

$$\begin{aligned}\mathcal{R}^{f}_{c,\,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} \;&=\; A_{rs}(c) \\ &\times\sum\overline{\rho^{(\pm)}_{|f|}(S_{-K}L_{-M}\chi_{rs}, w^{+}_{3}, w^{+}_{4})}\;\left[B^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\right]^{KM,LN}\rho^{(\pm)}_{|f|}(S_{-L}L_{-N}\chi_{rs}, w^{+}_{2}, w^{+}_{1}),\end{aligned} \tag{5.5}$$

with

$$A_{rs}(c) \;=\; \lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\chi^{\Delta}_{rs}|\chi^{\Delta}_{rs}\rangle}{\Delta-\Delta_{rs}(c)}\right)^{-1}. \tag{5.6}$$

The coefficients are the same as in the NS-NS sector and their explicit form has been conjectured in [21] to be:

$$A_{rs}(c) \;=\; \frac{1}{2}\prod_{p=1-r}^{r}\prod_{q=1-s}^{s}\left(\frac{1}{\sqrt{2}}\left(pb+\frac{q}{b}\right)\right)^{-1}, \quad p+q\in 2\mathbb{Z},\; (p,q)\neq(0,0),(r,s). \tag{5.7}$$

The factorization property of the forms $\rho^{(\pm)}_{\rm NR}$ holds in the present case only on singular vectors. In the case $\frac{rs}{2}\in\mathbb{N}$ one gets

$$\begin{aligned}
\rho^{(\pm)}_{\rm NRe}(S_{-I}\chi_{rs},w_2^+,w_1^+) &= \rho^{(\pm)}_{\rm NRe}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}},w_2^+,w_1^+)\,\rho^{(\pm)}_{\rm NRe}(\chi_{rs},w_2^+,w_1^+), \qquad |I|\in\mathbb{N},\\
\rho^{(\pm)}_{\rm NRo}(S_{-K}\chi_{rs},w_2^+,w_1^+) &= \rho^{(\pm)}_{\rm NRo}(S_{-K}\nu_{\Delta_{rs}+\frac{rs}{2}},w_2^+,w_1^+)\,\rho^{(\pm)}_{\rm NRe}(\chi_{rs},w_2^+,w_1^+), \quad |K|\in\mathbb{N}+\tfrac{1}{2},
\end{aligned}$$

while in the case $\frac{rs}{2}\in\mathbb{N}+\frac{1}{2}$

$$\begin{aligned}
\rho^{(\pm)}_{\rm NRo}(S_{-I}\chi_{rs},w_2^+,w_1^+) &= \rho^{(\mp)}_{\rm NRe}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}},w_2^+,w_1^+)\,\rho^{(\pm)}_{\rm NRo}(\chi_{rs},w_2^+,w_1^+), \qquad |I|\in\mathbb{N},\\
\rho^{(\pm)}_{\rm NRe}(S_{-K}\chi_{rs},w_2^+,w_1^+) &= i\,\rho^{(\mp)}_{\rm NRo}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}},w_2^+,w_1^+)\,\rho^{(\pm)}_{\rm NRo}(\chi_{rs},w_2^+,w_1^+), \quad |K|\in\mathbb{N}+\tfrac{1}{2}.
\end{aligned}$$

This yields

$$\begin{aligned}
&\mathcal{R}^{f}_{c,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} = \\
&\qquad A_{rs}(c)\,\overline{\rho^{(\pm)}_{\rm RNe}(\chi_{rs},w_3^+,w_4^+)}\,\rho^{(\pm)}_{\rm NRe}(\chi_{rs},w_2^+,w_1^+)F^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}
\end{aligned} \tag{5.8}$$

for $\frac{rs}{2}\in\mathbb{N}\cup\{0\}$, and

$$\begin{aligned}
&\mathcal{R}^{f}_{c,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} = \\
&\qquad A_{rs}(c)\,\overline{\rho^{(\pm)}_{\rm RNo}(\chi_{rs},w_3^+,w_4^+)}\,\rho^{(\pm)}_{\rm NRo}(\chi_{rs},w_2^+,w_1^+)F^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\begin{bmatrix}\mp\beta_3 & \mp\beta_2\\ \beta_4 & \beta_1\end{bmatrix}
\end{aligned} \tag{5.9}$$

for $\frac{rs}{2}\in\mathbb{N}-\frac{1}{2}$.

In order to calculate $\rho^{(\pm)}_{\rm NRp}(\chi_{rs},w_2^+,w_1^+)$ we shall first consider the three point correlation functions with degenerate field χ_{rs} within the Feigin-Fuchs construction [22]. In this approach the Ramond fields are represented by vertex operators in the free superscalar Hilbert space

$$R^{+}_{\beta,\bar\beta}(z,\bar z) = {\rm e}^{a\phi(z)+\bar a\bar\phi(\bar z)+i\frac{\pi}{4}}\sigma^{+}(z,\bar z), \qquad R^{-}_{\beta,\bar\beta}(z,\bar z) = {\rm e}^{a\phi(z)+\bar a\bar\phi(\bar z)-i\frac{\pi}{4}}\sigma^{-}(z,\bar z), \tag{5.10}$$

where $a=\frac{Q}{2}-\sqrt{2}\beta$ and $\sigma^{\pm}$ are the twist operators of the fermionic sector:

$$\psi(z)\sigma^{\pm}(w,\bar w) \sim \frac{1}{\sqrt{2(z-w)}}\,\sigma^{\mp}(w,\bar w). \tag{5.11}$$

The left chiral screening charges are given by:

$$Q_b = \oint dz\,\psi(z){\rm e}^{b\phi(z)}, \qquad Q_{\frac{1}{b}} = \oint dz\,\psi(z){\rm e}^{\frac{1}{b}\phi(z)},$$

and the same construction holds in the right sector. Let us consider the Feigin-Fuchs representation of three point functions with various number of left screening charges:

$$C^{\epsilon}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} = \left\langle \chi_{rs} R^{\epsilon}_{\beta_2} R^{\epsilon}_{\beta_1}\, Q_b^k\, Q^l_{\frac{1}{b}} \right\rangle, \qquad k+l\in 2\mathbb{N}\,, \qquad \delta = -\frac{1}{2\sqrt{2}}\left(\frac{1}{b}+b\right), \tag{5.12}$$

$$C^{\epsilon}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} = \left\langle \chi_{rs} R^{\epsilon}_{\beta_2} R^{\epsilon}_{\beta_1}\, Q_b^k\, Q^l_{\frac{1}{b}}\, \bar{Q}_b \right\rangle, \quad k+l\in 2\mathbb{N}+1, \quad \delta = \frac{1}{2\sqrt{2}}\left(\frac{1}{b}-b\right).$$

The charge conservation implies that the structure constants above are non-zero if and only if the even fusion rules ($k+l\in 2\mathbb{N}\cup\{0\}$):

$$\beta_1+\beta_2 \;=\; \frac{1}{2\sqrt{2}}\,(1-r+2k)b + \frac{1}{2\sqrt{2}}\,(1-s+2l)\frac{1}{b}, \tag{5.13}$$

or the odd fusion rules ($k+l\in 2\mathbb{N}-1$):

$$\beta_1+\beta_2 \;=\; \frac{1}{2\sqrt{2}}\,(1-r+2k)b + \frac{1}{2\sqrt{2}}\,(1-s+2l)\frac{1}{b} \tag{5.14}$$

are satisfied (k,l are integers in the range $0\le k\le r-1,\ 0\le l\le s-1$).

In the Feigin-Fuchs representation one can show that for any even integer $n\in 2\mathbb{N}$:

$$\begin{aligned}\left\langle \psi(w_1)\ldots\psi(w_n)\sigma^-(1,1)\sigma^-(0,0)\right\rangle &= -\left\langle \psi(w_1)\ldots\psi(w_n))\sigma^+(1,1)\sigma^+(0,0)\right\rangle,\\ \left\langle \psi(w_1)\ldots\psi(w_{n-1})\,\bar{\psi}(\bar{w})\sigma^-(1,1)\sigma^-(0,0)\right\rangle &= \left\langle \psi(w_1)\ldots\psi(w_{n-1})\,\bar{\psi}(\bar{w})\sigma^+(1,1)\sigma^+(0,0)\right\rangle.\end{aligned}$$

If the even fusion rules (5.13) are satisfied this implies

$$C^{+}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} = -C^{-}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)}.$$

It follows that $C^{(-)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} \neq 0$ and therefore the corresponding form has to vanish:

$$\rho^{(-)}_{\mathrm{NRe,o}}(\chi_{rs}, w_2^+, w_1^+) = 0.$$

Similarly, for the odd fusion rules (5.14) one gets $C^{(+)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} \neq 0$ and

$$\rho^{(+)}_{\mathrm{NRe,o}}(\chi_{rs}, w_2^+, w_1^+) = 0.$$

An additional information on zeros of the forms in question can be derived from the formula

$$C^{(\pm)}_{(\alpha_{rs},\delta),(-\beta_2,0),(\beta_1,0)} = C^{(\mp)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)},$$

which is a simple consequence of (4.14). The form $\rho^{(+)}_{\mathrm{NRe,o}}(\chi_{rs}, w_2^+, w_1^+)$ has to vanish for the even fusion rules (5.13) and $\rho^{(+)}_{\mathrm{NRe,o}}(\chi_{rs}, w_2^+, w_1^+)$ for the odd fusion rules (5.14) with the opposite sign in front of β_2 in both cases.

The discussion above suggests the following definition of the fusion polynomials in the Ramond sector:

$$P^{rs}_c\left[{}^{\pm\beta_2}_{\;\beta_1}\right] \;=\; \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\beta_1\mp\beta_2+\frac{pb+qb^{-1}}{2\sqrt{2}}\right)\prod_{p'=1-r}^{r-1}\prod_{q'=1-s}^{s-1}\left(\beta_1\pm\beta_2+\frac{p'b+q'b^{-1}}{2\sqrt{2}}\right) \tag{5.15}$$

where p, q, p', q' run with the step 2 and satisfy the conditions: $p + q - (r + s) \in 4\mathbb{Z} + 2$ and $p' + q' - (r + s) \in 4\mathbb{Z}$. One easily checks that for $rs \in 2\mathbb{N}$, $P_c^{rs}\left[\begin{smallmatrix}\pm\beta_2\\ \beta_1\end{smallmatrix}\right]$ are polynomials of degree $\frac{rs}{2}$ in $(\Delta_2 - \Delta_1)$, and for $rs \in 2\mathbb{N} - 1$ — of degree $\frac{rs-1}{2}$ in $(\Delta_2 - \Delta_1)$ with the additional factor $(\beta_1 \mp \beta_2)$. On the other hand

$$\begin{aligned}\rho^{(\pm)}_{\mathrm{NRe}}(L_{-1}^n\nu, w_2^+, w_1^+) &= (\Delta + \Delta_2 - \Delta_1)_n\\ \rho^{(\pm)}_{\mathrm{NRo}}(S_{-\frac{1}{2}}L_{-1}^n\nu, w_2^+, w_1^+) &= \mathrm{e}^{-i\frac{\pi}{4}}(\beta_1 \mp \beta_2)(\Delta + \Delta_2 - \Delta_1)_n\end{aligned}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. Taking into account our normalization condition for χ_{rs} one thus finally gets

$$\begin{aligned}\rho^{(\pm)}_{\mathrm{NRe}}(\chi_{rs}, w_2^+, w_1^+) &= P_c^{rs}\left[\begin{smallmatrix}\pm\beta_2\\ \beta_1\end{smallmatrix}\right] &&\text{for } rs \in 2\mathbb{N},\\ \rho^{(\pm)}_{\mathrm{NRo}}(\chi_{rs}, w_2^+, w_1^+) &= \mathrm{e}^{-i\frac{\pi}{4}}\, P_c^{rs}\left[\begin{smallmatrix}\pm\beta_2\\ \beta_1\end{smallmatrix}\right] &&\text{for } rs \in 2\mathbb{N} - 1.\end{aligned} \tag{5.16}$$

6. Elliptic recurrence

As in the bosonic and the NS cases the first step in a derivation of the elliptic recurrence is to find the large Δ asymptotic of the conformal block. The method of calculations proposed in [4] is based on the observation that the full dependence of the first two terms in the large Δ expansion on the variables Δ_i, c can be read off from the first two terms of the $\frac{1}{\delta}$ expansion of the classical block. The essential point of this approach is the existence of the classical limit of conformal blocks. In the present case this limit is to some extend justified by the path integral representation of the $N = 1$ super-Liouville amplitudes defined by the action:

$$\mathcal{S}_{\mathrm{SLFT}} = \int d^2z \left(\frac{1}{2\pi}\,|\partial\phi|^2 + \frac{1}{2\pi}\left(\psi\bar\partial\psi + \bar\psi\partial\bar\psi\right) + 2i\mu b^2\bar\psi\psi\mathrm{e}^{b\phi} + 2\pi b^2\mu^2\mathrm{e}^{2b\phi}\right). \tag{6.1}$$

Within the functional approach the Ramond fields are represented by vertex operators (5.10). Since the twist fields are light the fermionic sector does not contribute to the classical limit at all. Strictly speaking the path integral representation implies the classical limit of the whole amplitude. Imposing extra restrictions on intermediate states one may extend the argument to the sum of bilinear block products with fixed parity and intermediate weight. Considering various amplitudes with the same classical limit one may get the information concerning individual terms. This leads to the assumption that in the limit

$$b \to 0, \qquad ib\beta_i \to p_i, \qquad b^2\Delta_i \to \delta_i = p_i^2,$$

the asymptotic behavior of each conformal block takes the form

$$\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) \sim r_1\mathrm{e}^{\frac{1}{2b^2}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \qquad \mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) \sim r_{\frac{1}{2}}\mathrm{e}^{\frac{1}{2b^2}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \tag{6.2}$$

where $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$ is the classical conformal block and coefficients $r_1, r_{\frac{1}{2}}$ are independent of b. Analyzing the leading powers of Δ in the forms $\rho^{+-}_{NR}, \rho^{-+}_{NR}$

$$\begin{aligned}\rho^{+-}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\propto \beta_1\Delta^{\frac{|K|-1}{2}} + \dots,\\ \rho^{-+}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\propto \beta_2\Delta^{\frac{|K|-1}{2}} + \dots,\end{aligned}$$

and the Δ dependence of the Gram matrix one can show that

$$r_1 \propto \text{const}, \qquad r_{\frac{1}{2}} \propto \frac{1}{\delta},$$

as functions of δ.

Once the classical limits are known one can follow Zamolodchikov's derivation in order to find large Δ behavior. In the present case it yields:

$$\ln \mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) = \pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z)$$
$$+\left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \Delta_2\right)\ln(z) + f^{\pm\pm}(z) + \mathcal{O}\left(\frac{1}{\Delta}\right), \quad (6.3)$$
$$\ln \mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) = -\ln\Delta + \pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z)$$
$$+\left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \Delta_2\right)\ln(z) + \mathcal{O}\left(\frac{1}{\Delta}\right), \quad (6.4)$$

where

$$\tau = i\frac{K(1-z)}{K(z)}$$

and $f^{\pm\pm}(z)$ are functions of z specific for each type of block and independent of Δ_i and c. The large Δ asymptotic suggests the following form of superconformal blocks:

$$\mathcal{F}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) = (16q)^{\Delta - \frac{c-3/2}{24}}\, z^{\frac{c-3/2}{24} - \Delta_1 - \Delta_2}\, (1-z)^{\frac{c-3/2}{24} - \Delta_2 - \Delta_3} \quad (6.5)$$
$$\times \theta_3^{\frac{c-3/2}{2} - 4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\, \mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q),$$

where $q = \exp(i\pi\tau)$. The elliptic blocks $\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)$ have the same analytic structure as superconformal ones:

$$\mathcal{H}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q) = g^{\pm\pm}(q) + \sum_{m,n} \frac{h^{1}_{mn}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)}{\Delta - \Delta_{mn}},$$
$$\mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q) = \sum_{m,n} \frac{h^{\frac{1}{2}}_{mn}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)}{\Delta - \Delta_{mn}}.$$

The functions $g^{\pm\pm}(q)$ depend on the block type and are independent of the external momenta β_i and the central charge c. They have no singularities in Δ and are directly related to the functions $f^{\pm\pm}(z)$ in (6.3). The analytic form of these functions can be read off from the $\hat{c} = 1$ elliptic blocks with $\Delta_i = \Delta_0 = \frac{1}{16}$. The explicit formula for this blocks can be obtained using the techniques of the chiral superscalar model [17]. The chiral correlation function projected on the intermediate Δ NS module of the fields σ_0 corresponding to the lowest Ramond state in the bosonic sector and the NS vacuum state in the fermionic sector takes the form:

$$\left\langle \sigma_0 \middle| \sigma_0(1)\middle|_{\Delta} \sigma_0(z) \middle| \sigma_0 \right\rangle = \left|\left\langle \nu_\Delta \middle| \sigma_0(1) \middle| \sigma_0 \right\rangle\right|^2 (16q)^{\Delta}\, [z(1-z)]^{-\frac{1}{8}}\, \theta_3(q)^{-1}. \quad (6.6)$$

On the other hand the $b \to i$, $\beta_i \to 0$ limit of each type of general even block is regular for generic values of Δ and yields

$$\lim_{\beta\to 0}\lim_{b\to i}\mathcal{F}^1_\Delta\left[\begin{smallmatrix}\pm\beta & \pm\beta\\ \beta & \beta\end{smallmatrix}\right](z) = (16q)^\Delta\ [z(1-z)]^{-\frac{1}{8}}\ \theta_3(q)^{-1}g^{\pm\pm}(q)\ .$$

Comparing with (6.6) one gets

$$g^{\pm\pm}(q) = 1$$

which completes our derivation of the elliptic recurrence in the Ramond sector.

Acknowledgments

The work was supported by the Polish State Research Committee (KBN) grant no. N N202 0859 33.

P.S. is grateful to the faculty of the Institute of Theoretical Physics, University of Wrocław for the hospitality.

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